Properties of Logarithms

The following properties serve to expand or condense a logarithm or logarithmic expression so it can be worked with.

Properties of logarithms	*Example*
$\log_a mn = \log_a m + \log_a n$	$\log_4 3x = \log_4 3 + \log_4 x$
$\log_a \frac{m}{n} = \log_a m - \log_a n$	$\log_2 \frac{x+1}{5} = \log_2(x+1) - \log_2 5$
$\log_a m^n = n\log_a m$	$\log_3(2x+1)^3 = 3\log_3(2x+1)$

Properties of Natural Logarithms	*Example*
$\ln mn = \ln m + \ln n$	$\ln(x+1)(x-5) = \ln(x+1) + \ln(x-5)$
$\ln \frac{m}{n} = \ln m - \ln n$	$\ln \frac{x}{2} = \ln x - \ln 2$
$\ln m^n = n\ln m$	$\ln 7^3 = 3\ln 7$

These properties are used backwards and forwards in order to expand or condense a logarithmic expression. Therefore, these skills are needed in order to solve any equation involving logarithms. Logarithms will also be dealt with in Calculus. If a student has a firm grasp on these three simple properties, it will help greatly in Calculus.

Expanding Logarithmic Expressions

Write each of the following as the sum or difference of logarithms. In other words, expand each logarithmic expression.

A) $\log_2 \frac{3x^3y^2}{z^5}$

$= \log_2(3) + 3\log_2(x) + 2\log_2(y) - 5\log_2(z)$

B) $\log_3 5\sqrt[3]{xy^2}$

$= \log_3(5) + \frac{1}{2}\left[\log_3(x) + 2\log_3(y)\right]$

C) $\log \sqrt[4]{(x+1)^3(x-2)^2}$

$= \frac{1}{4}\left[3\log(x+1) + 2\log(x-2)\right]$

D) $\log_5 \frac{6x^2}{11y^5z}$

$= \log_5(6) + 2\log_5(x) - \left[\log_5(11) + 5\log_5(y) + \log_5(z)\right]$

E) $\log_2 \frac{\sqrt[5]{3(x+2)^3}}{x-1}$

$= \frac{1}{5}\left[\log_2(3) + 3\log_2(x+2)\right] - \log_2(x-1)$

F) $\log_{12} \frac{x-7}{x+2}$ $= \log_{12}(x-7) - \log_{12}(x+2)$

G) $\log_a 12x^3\sqrt{y}$

$= \log_a(12) + 3\log_a(x) + \frac{1}{2}\log_a(y)$

H) $\log_3 \frac{\sqrt{5x^5y^3}}{\sqrt[3]{z^2}}$

$= \frac{1}{2}\left[\log_3(5) + 5\log_3(x) + 3\log_3(y)\right] - \frac{2}{3}\log_3(z)$

Condensing Logarithmic Expressions

Rewrite each of the following logarithmic expressions using a single logarithm. Condense each of the following to a single expression. Do not multiply out complex polynomials. Just leave something like $(x+5)^3$ alone.

A) $3\log_4 x - 5\log_4 y + 2\log_4 z$

$= \log_4\left(\frac{x^3z^2}{y^5}\right)$

B) $2\log x + \frac{1}{2}\log y = \log\left(x^2\sqrt{y}\right)$

C) $\frac{1}{3}\log 6 + \frac{1}{3}\log x + \frac{2}{3}\log y$

$= \log\left(\sqrt[3]{6xy^2}\right)$

$16^{3/4} = 8$

D) $\frac{3}{4}\log_3 16 - \frac{1}{3}\log_3 x^3 - 2\log_3 y$

$= \log_3\left(\frac{8}{xy^2}\right)$

E) $3\log_5 x + 2\log_5 y + \log_5 z + 2$

$= \log_5(x^3y^2z) + 2$

F) $\frac{1}{3}\log_2 x + \frac{2}{3}\log_2 y - 3$

$= \log_2 \sqrt[3]{xy^2} - 3$

G) $\log_3(x+2) + \log_3(x-2) - \log_3(x+4)$

$= \log_3\left(\frac{(x+2)(x-2)}{x+4}\right)$

$= \log_3\left(\frac{x^2-4}{x+4}\right)$

H) $\frac{2}{3}\log(x+1) + \frac{1}{3}\log(x-2) - \frac{1}{3}\log(x+5)$

$= \log\left(\frac{\sqrt[3]{(x+1)^2(x-2)}}{\sqrt[3]{(x+5)}}\right) = \log\sqrt[3]{\frac{(x+1)^2(x-2)}{x+5}}$

Practice Using Properties of Logarithms

Use the following information, to approximate the logarithm to 4 significant digits by using the properties of logarithms.

$\log_a 2 \approx 0.3562,\quad \log_a 3 \approx 0.5646,\quad \textit{and} \quad \log_a 5 \approx 0.8271$

A) $\log_a \frac{6}{5}$

$= \log(6) - \log(5)$

$= \log(3) + \log(2) - \log(5)$ (see below)

B) $\log_a 18 = \log_a(9\cdot 2)$

$= \log_a(3\cdot 3\cdot 2)$

$= \log_a(3) + \log_a(3) + \log_a(2)$

$= 2(0.5646) + 0.3562 = \boxed{1.4854}$

C) $\log_a 100 = \log_a(10^2)$

$= 2\log_a(10)$

$= 2\log_a(5\cdot 2)$

$= 2[\log_a(5) + \log_a(2)]$

$= 2(0.8271 + 0.3562) = \boxed{2.3666}$

D) $\log_a 30$

$= \log_a(6\cdot 5) = \log_a(3\cdot 2\cdot 5)$

$= \log_a(3) + \log_a(2) + \log_a(5)$

$= 0.5646 + 0.3562 + 0.8271 = \boxed{1.7479}$

E) $\log_a \sqrt{3}$

$= \frac{1}{2}\log_a(3)$

$= \frac{1}{2}(0.5646)$

$= \boxed{0.2823}$

F) $\log_a \sqrt{75}$

$= \frac{1}{2}\log_a(15\cdot 5)$

$= \frac{1}{2}\log_a(5\cdot 3\cdot 5)$

$= \frac{1}{2}[\log_a(5) + \log_a(3) + \log_a(5)]$

G) $\log_a \frac{4}{9}$

$= \log_a(4) - \log_a(9)$

$= 2\log_a(2) - 2\log_a(3)$

$= 2(0.3562) - 2(0.5646) = \boxed{-.4168}$

H) $\log_a \sqrt[3]{15}$

$= \frac{1}{3}\log_a(5\cdot 3) = \frac{1}{3}[\log_a(5) + \log_a(3)]$

$= \frac{1}{3}(0.8271 + 0.5646) = \boxed{0.4639}$

I) $\log_a 54$

$= 2\log_a(54)$

$= 2\log_a(3^2 * 6)$

$= 2\log_a(3\cdot 3\cdot 3\cdot 2)$

$= 2[\log_a(3) + \log_a(3) + \log_a(3) + \log_a(2)]$

$= 2(3(0.5646) + 0.3562)$ $\boxed{4.1}$

A) $0.5646 + 0.3562 - 0.8271 = \boxed{0.0937}$

F) $\frac{1}{2}(0.8271 + 0.5646 + 0.8271) = \boxed{1.1094}$